Annals of Finance (2007) 3:471–486
DOI 10.1007/s10436-006-0055-x

RESEARCH ARTICLE

Duration, factor sensitivities, and interest rate Greeks

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Received: 9 June 2006 / Accepted: 15 September 2006 / Published online: 27 October 2006

Abstract In this paper, we introduce for interest rate sensitive assets the natural analogs of delta and gamma for equity options by considering the derivatives of asset prices with respect to the directions along which the forward rate curve may evolve. Macaulay duration and convexity, as well as stochastic duration considered in Cox et al. (J Business 52:51–61, 1979) and Munk (Rev Derivat Res 3:157–181, 1999), are easily obtained as special cases of these in which the derivatives are computed along parallel shifts and the direction of the forward rate volatilities, respectively. Moreover, we demonstrate using the example of the Ritchken and Sankarasubramanian (Math Financ 5:55–72, 1995) model that the hedging strategy based on these sensitivity measures provides a superior performance in comparison to the traditional duration based hedging approaches.

Keywords Interest rate risk · Duration · Stochastic duration · Factor sensitivities

JEL Classification Numbers E43 · G11 · G12

1 Introduction

For stock options within the Black and Scholes (1973) framework, option Greeks are the obvious measures of price sensitivity, and consequently they are used for hedging and risk management purposes. For interest rate derivatives,

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however, there does not appear to be an obvious counterpart except in the special cases where similar pricing formulae are available.

Duration, as introduced in Macaulay (1938), has traditionally been used as a measure of interest rate risk, especially for bonds. But as pointed out in Cox et al. (1979), duration is appropriate for this purpose only for parallel shifts in the yield curve. An examination of the way in which duration for an asset is defined in terms of the derivative of its price with respect to the yield, however, suggests that what one needs to consider is the derivative of the asset price with respect to the *entire* yield curve in some sense. In the special case where the entire yield curve is a deterministic function of the short rate, such as in the Cox et al. (1985) model, differentiation with respect to the entire yield curve is equivalent to differentiation with respect to the short rate, and this is precisely the approach taken in Cox et al. (1979) to define *stochastic duration*. Since it is unclear which derivative should be taken when there are multiple factors, an alternative definition using zero coupon bond price volatilities was introduced in Munk (1999).

If we consider movements in the forward rate curve in the absence of a model for the term structure of interest rates, then the forward rate curve may evolve in an arbitrary manner, and we must then consider all possible movements when assessing interest rate risk. The delta and gamma introduced in this paper provides a measure of sensitivity for asset prices along any given direction of movement in the forward rate curve. If a term structure model is assumed, then the movements in the forward rate curve are constrained by the assumed dynamics, and the directions for movement in the forward rate curve are limited. If, moreover, the model admits a finite-dimensional realization (FDR), in the sense of Björk and Svensson (2001), then the space of possible movements in the forward rate curve is finite dimensional, and the task of hedging against such movements becomes much more manageable.

It is shown in this paper that the notion of duration introduced in Macaulay (1938) is essentially a special case of the delta introduced in this paper corresponding to parallel shifts in the forward rate curve, and the stochastic duration introduced in Cox et al. (1979) is a special case corresponding to movements along the direction of forward rate volatility under the Cox et al. (1985) model. Geometric considerations establish that these quantities are insufficient to fully capture the risks associated with movements in the forward rate curve, and are, in some cases, even inconsistent with the underlying term structure model. For example, parallel shifts in the forward rate curves are *not* possible under most term structure models,[1] and so Macaulay duration, which measures sensitivity to such movements, is inconsistent with these models.

Moreover, most of the hedging strategies for interest rate sensitive instruments only consider movements along the direction of the forward rate volatilities. Results from Björk and Svensson (2001) and Filipović and Teichmann (2004), however, show that these are *not* the only directions in which the forward rate curves may move. In general, the space of directions for forward rate

[1] It is noted that the Ho and Lee (1986) model is an example of a term structure model consistent with parallel shifts in the forward rate curve.

curve movements is the Lie algebra generated by the forward rate volatilities and the Stratonovich forward rate drift, considered as vector fields on a suitable space of curves. To fully hedge against all possible movements in the forward rate curve, one must first compute this Lie algebra, and then consider the sensitivities along directions corresponding to a basis of this Lie algebra. We demonstrate the benefits of considering directional derivatives by comparing the resulting hedging strategy against those commonly used in practice.

The remainder of this paper is organized as follows. The directional delta and gamma are introduced in Sect. 2, and computed for the zero coupon bonds. The notion of directional duration is introduced in Sect. 3 and shown to be the natural generalization of the Macaulay duration. In Sect. 4, delta and gamma for term structure models with FDR is considered. It is shown that stochastic duration introduced in Cox et al. (1979) is a special case of the directional duration, and that the former is an inadequate measure of interest rate risk. It is also shown that a simple adaptation of delta hedging for stock options to interest rate derivatives does not provide a complete hedge against movements in the forward rate curve. Brief discussion on the conditions under which delta and gamma can be defined in general is given in Sect. 5, and a numerical investigation of the directional delta hedging strategy is considered in Sect. 6. The paper concludes with Sect. 7.

2 Interest rate delta and gamma

In this section, we introduce the notion of *delta* and *gamma* for interest rate sensitive assets with respect to movements in the forward rate curve. The definition does not rely on, nor does it assume, the existence of an underlying model for the term structure of interest rates. That is, we do not impose any a priori restrictions on the possible movements in the forward rate curve. The situation in which the evolution of the forward rate curve is determined by a term structure model will be considered in Sect. 4.

In order to define the notion of interest rate delta and gamma with respect to as wide a class of forward rate curves possible, we only assume that the instantaneous forward rate curves are continuous. Hence, we take for the set of admissible forward rate curves, $\mathscr{A} = C(\mathbb{R}_+)$. Then arguments analogous to Rudin (1991), Example 1.44, shows that $\mathscr{A}$ is, in fact, a Fréchet space. We provide a brief review of this, since it establishes concrete characterizations of useful objects such as bounded subsets of $\mathscr{A}$. Recall that a subset B of a topological vector space X is *bounded* if for every neighborhood U of $0 \in X$ there exists $\alpha \in \mathbb{R}_+$ such that $B \subset xU$ for all $x \geq \alpha$, and a map $\phi\colon X \to Y$ between topological vector spaces X and Y is *bounded* if $\phi(B)$ is bounded for every bounded $B \subset X$.

For each $n \in \mathbb{N}$, let $K_n = [0, n] \subset \mathbb{R}_+$, and define $\rho_n\colon \mathscr{A} \to \mathbb{R}_+$ by

$$\rho_n(f) = \sup\{|f(x)|\colon x \in K_n\}. \tag{1}$$

Then ρ_n is a seminorm for all $n \in \mathbb{N}$ and $\{\rho_n : n \in \mathbb{N}\}$ is a sequence of separating seminorms that topologize $\mathscr{A}$. The local base corresponding to the resulting topology is given by $\{V_n : n \in \mathbb{N}\}$, where

$$V_n = \left\{ f \in \mathscr{A} : \rho_n(f) < \frac{1}{n} \right\}. \tag{2}$$

The sets V_n are convex and so $\mathscr{A}$ is locally convex. A subset $\mathscr{B} \subset \mathscr{A}$ is bounded if and only if there exists a sequence $\{\alpha_n\} \subset \mathbb{R}_+$ such that $\rho_n(f) < \alpha_n$ for all $f \in \mathscr{B}$ and $n \in \mathbb{N}$. The topology is compatible with the metric, d, on $\mathscr{A}$ given by

$$d(f,g) = \max_{n\in\mathbb{N}} \frac{2^{-n}\rho_n(f-g)}{1+\rho_n(f-g)}. \tag{3}$$

In particular, $\mathscr{A}$ is metrizable. Moreover, $\mathscr{A}$ is complete, and since d is invariant, $\mathscr{A}$ is a Fréchet space.

Let $\mathscr{U} \subset \mathscr{A}$ be open and let $\phi : \mathscr{U} \to \mathbb{R}$. Then ϕ is said to be Gáteaux differentiable on $\mathscr{U}$ if the limit,

$$\partial_f\phi(v) = \lim_{t\to 0} \frac{\phi(f+tv) - \phi(f)}{t}, \tag{4}$$

exists for all $f \in \mathscr{U}$ and $v \in \mathscr{A}$, and the map $\partial_{\cdot}\phi : \mathscr{U} \times \mathscr{A} \to \mathbb{R}$ is continuous. The set of Gáteaux differentiable maps on $\mathscr{U}$ will be denoted $C^1(\mathscr{U})$. A map $\phi \in C^1(\mathscr{U})$ is said to be twice Gáteaux differentiable on $\mathscr{U}$ if the limit,

$$\partial_f^2\phi(u,v) = \lim_{t\to 0} \frac{\partial_{f+tv}\phi(u) - \partial_f\phi(u)}{t}, \tag{5}$$

exists for all $f \in \mathscr{U}$ and $u, v \in \mathscr{A}$, and the map $\partial_{\cdot}^2\phi : \mathscr{U} \times \mathscr{A} \times \mathscr{A} \to \mathbb{R}$ is continuous. The space of all such maps will be denoted $C^2(\mathscr{U})$. For further details, refer to Kriegel and Michor (1997), p. 128, or Abraham et al. (1988), Chap. 2. Although it is possible to define differentiability of higher orders, we will only focus on the first two since they are of primary concern in financial applications.

The elements of $C^2(\mathscr{U})$ will be considered as the space of interest rate sensitive assets on $\mathscr{U}$. We now consider the simplest example of such an asset. For any $x \in \mathbb{R}_+$, define the linear mapping $I_x : \mathscr{A} \to \mathbb{R}$ by

$$I_x(f) = \int_0^x f(s)\,\mathrm{d}s. \tag{6}$$

Note that since f is continuous, I_x is well-defined for all $x \in \mathbb{R}_+$.

Lemma 1 *For each $x \in \mathbb{R}_+$ the map $I_x : \mathscr{A} \to \mathbb{R}$ is bounded. In particular, I_x is continuous.*

Proof Let $\mathscr{B} \subset \mathscr{A}$ be bounded. Then there exists a sequence $\{\alpha_n\} \subset \mathbb{R}_+$ such that $\rho_n(f) < \alpha_n$ for all $f \in \mathscr{B}$ and $n \in \mathbb{N}$. Now, let $n_x = \min\{n \in \mathbb{N} \colon x \leq n\}$. Then for any $f \in \mathscr{B}$, we have $|I_x(f)| \leq x\rho_{n_x}(f) \leq x\alpha_{n_x}$, and so $I_x(\mathscr{B})$ is bounded. Since a linear map from a metrizable topological space to a topological vector space is bounded if and only if it is continuous by Rudin (1991), Theorem 1.32, it follows that I_x is continuous. □

Lemma 2 *For all $x \in \mathbb{R}_+$, we have $I_x \in C^2(\mathscr{A})$. Moreover, $\partial_f I_x = I_x$ and $\partial_f^2 I_x = 0$.*

Proof Let $x \in \mathbb{R}_+$ and $f \in \mathscr{A}$. Then since I_x is linear, $I_x(f + tv) = I_x(f) + tI_x(v)$ for any $v \in \mathscr{A}$ and $t \in \mathbb{R}$, and so

$$\frac{I_x(f+tv) - I_x(f)}{t} = I_x(v)$$

whence $\partial_f I_x = I_x$. It follows from Lemma 1 that I_x is Gâteaux differentiable at f. The fact that $\partial_f^2 I_x = 0$ is an immediate consequence of the linearity of I_x. □

For any $x \in \mathbb{R}_+$, define the x-maturity zero coupon bond, p_x, as the map $\mathscr{A} \to \mathbb{R}$ given by

$$p_x(f) = \mathrm{e}^{-I_x(f)}. \tag{7}$$

Since the exponential map $\mathrm{e} \colon \mathbb{R} \to \mathbb{R}$ is bounded,[2] it follows from the previous lemma that $p_x = \mathrm{e} \circ (-I_x) \colon \mathscr{A} \to \mathbb{R}$ is bounded for all $x \in \mathbb{R}_+$.

Corollary 1 *We have $p_x \in C^2(\mathscr{A})$ for all $x \in \mathbb{R}_+$. Moreover, for $k \in \{1, 2\}$,*

$$\partial_f^k p_x = (-1)^k p_x(f) I_x^k, \tag{8}$$

where $I_x^2 \colon \mathscr{A} \times \mathscr{A} \to \mathbb{R}$ is given by $I_x^2(v_1, v_2) = I_x(v_1)I_x(v_2)$.

Proof The fact that $p_x \in C^2(\mathscr{A})$ follows from Lemma 2, since $p_x = \mathrm{e} \circ (-I_x)$ and the exponential map is differentiable. The expression for $\partial_f^k p_x$ follows from the chain rule. □

Note that setting $k = 1$ in (8) gives $\partial_f p_x(v) = -p_x(f)I_x(v)$, which coincides with the intuitive notion of the change in p_x if the forward rate curve shifts from f to $f + v$.

Definition 1 *Let $\mathscr{U} \subset \mathscr{A}$ be open and let $\phi \in C^2(\mathscr{U})$. Then for any $f \in \mathscr{U}$ and $u, v \in \mathscr{A}$, the quantities $\partial_f \phi(v)$ and $\partial_f^2 \phi(u, v)$ will be called the* delta *in the direction of v and the* gamma *in the direction (u, v), respectively, of ϕ at f.*

[2] Considered as a map between topological vector spaces.

So given a forward rate curve, r, and a twice differentiable ϕ representing the price of an interest rate sensitive asset, we may consider the delta of ϕ along any direction $v \in \mathscr{A}$ and the gamma of ϕ along any pair of directions $(u, v) \in \mathscr{A}^2$. If a term structure model is assumed, then the directions to consider would be those consistent with the assumed model, while in the absence of a term structure model the directions may, for example, be those obtained as principal components.

It will be shown in the following sections that, apart from providing a more *complete* measure of sensitivity to movements in the forward rate curve, delta and gamma defined above naturally generalize the notions of duration and convexity, sensitivities along principal components, and option delta and gamma.

3 Connection to Macaulay duration and convexity

In this section, we explore the relationship between the delta and the gamma of $\phi \in C^2(\mathscr{U})$, defined in the previous section, and the duration and convexity defined in Macaulay (1938).

Let $m \in \mathbb{N}$, and for $\boldsymbol{c} = (c_1, \ldots, c_m) \in \mathbb{R}^m$ and $\boldsymbol{x} = (x_1 < \cdots < x_m) \in \mathbb{R}^m_+$, define a map $\phi_{\boldsymbol{c},\boldsymbol{x}} \in C^2(\mathscr{A})$ by

$$\phi_{\boldsymbol{c},\boldsymbol{x}}(f) = \sum_{i=1}^{m} c_i p_{x_i}(f). \tag{9}$$

The map $\phi_{\boldsymbol{c},\boldsymbol{x}}$ represents the price of an asset with cashflow, c_i, occurring at time, x_i, for $1 \leq i \leq m$, such as a coupon bond. Let $f \in \mathscr{A}$ such that $\phi_{\boldsymbol{c},\boldsymbol{x}}(f) \neq 0$. Then the duration and convexity, denoted $D^1_{\mathrm{Mac}}\phi_{\boldsymbol{c},\boldsymbol{x}}(f)$ and $D^2_{\mathrm{Mac}}\phi_{\boldsymbol{c},\boldsymbol{x}}(f)$, respectively, of $\phi_{\boldsymbol{c},\boldsymbol{x}}$ at f, as defined in Macaulay (1938), are

$$D^k_{\mathrm{Mac}}\phi_{\boldsymbol{c},\boldsymbol{x}}(f) = \frac{1}{\phi_{\boldsymbol{c},\boldsymbol{x}}(f)} \sum_{i=1}^{m} x_i^k c_i p_{x_i}(f) \tag{10}$$

for $k = 1, 2$. We now give an alternative, more general, definitions of directional duration and convexity.

Definition 2 *For any $\phi \in C^2(\mathscr{A})$ and $f, u, v \in \mathscr{A}$, define the duration in the direction v and the convexity in the direction (u, v) of ϕ at f, denoted $D^1_f\phi(v)$ and $D^2_f\phi(u, v)$, respectively, by*

$$D^1_f\phi(v) = \begin{cases} 0, & \textit{if } \partial_f\phi(v) = 0, \\ \pm\infty, & \textit{if } \phi(f) = 0 \textit{ and } \partial_f\phi(v) \lessgtr 0, \\ -\dfrac{\partial_f\phi(v)}{\phi(f)}, & \textit{if } \phi(f) \neq 0. \end{cases}$$
$$D^2_f\phi(u,v) = \begin{cases} 0, & \textit{if } \partial^2_f\phi(u,v) = 0, \\ \pm\infty, & \textit{if } \phi(f) = 0 \textit{ and } \partial^2_f\phi(v) \gtrless 0, \\ \dfrac{\partial^2_f\phi(u,v)}{\phi(f)}, & \textit{if } \phi(f) \neq 0. \end{cases} \tag{11}$$

It is now shown that directional duration and convexity are natural generalizations of the Macaulay duration and convexity. For notational convenience, let $D^2_f\phi(v) = D^2_f\phi(v,v)$ and $I^2_x(v) = I^2_x(v,v)$ for $x \in \mathbb{R}_+$ and $f, v \in \mathscr{A}$.

Proposition 1 *Let $k \in \{1,2\}$ and $f, v \in \mathscr{A}$. Then the directional duration and convexity defined above have the following properties:*

(a) $D^k_f p_x(v) = I^k_x(v)$ *for all* $x \in \mathbb{R}_+$. *In particular, if* $1 \in \mathscr{A}$ *is the constant map, then* $D^k_f p_x(1) = x^k$ *for all* $x \in \mathbb{R}_+$.

(b) *Let* $m \in \mathbb{N}$, $\boldsymbol{c} \in \mathbb{R}^m$ *and* $\boldsymbol{x} \in \mathbb{R}^m_+$ *such that* $\phi_{\boldsymbol{c},\boldsymbol{x}}(f) \neq 0$. *Then*

$$D^k_f\phi_{\boldsymbol{c},\boldsymbol{x}}(v) = \frac{1}{\phi_{\boldsymbol{c},\boldsymbol{x}}(f)} \sum_{i=1}^{m} I^k_{x_i}(v) c_i p_{x_i}(f). \tag{12}$$

In particular, $D^k_f\phi_{\boldsymbol{c},\boldsymbol{x}}(1) = (-1)^k D^k_{\mathrm{Mac}}\phi_{\boldsymbol{c},\boldsymbol{x}}(f)$.

Proof Immediate from definitions. □

Hence, Macaulay duration and convexity at $f \in \mathscr{A}$ corresponds to the duration and convexity at f as defined above in the direction $v = 1$, and it follows that Macaulay duration and convexity only consider the sensitivities of $\phi \in C^2(\mathscr{A})$ under *parallel shifts* of the forward rate curve. In the absence of a term structure model, however, one must consider the sensitivity of ϕ with respect to *all* directions $v \in \mathscr{A}$. Note that this implies, in particular, that it is *not* possible to *completely* hedge away risks associated with movements in the forward rate curve, unless some restrictions are imposed on the way the forward rate curve evolves over time.

If, however, one assumes that the forward rate curve evolves according to a specific term structure model, such as the Cox et al. (1985) model, then $1 \in \mathscr{A}$ may *not* be a direction that is consistent with the model, in which case Macaulay duration and convexity measure sensitivities with respect to movements that are *not permitted* under the assumed term structure model.

Definition 3 *Let $\phi \in C^1(\mathscr{A})$ and $f, u, v \in \mathscr{A}$. Then ϕ is said to be delta neutral at f in the direction v if $\partial_f\phi(v) = 0$. If $\phi \in C^2(\mathscr{A})$, then ϕ is said to be gamma neutral at f in the direction (u, v) if $\partial^2_f\phi(u,v) = 0$.*

This is a direct analog of the corresponding notions for bonds and options. We will consider delta and gamma neutral hedging strategies in subsequent sections.

4 Connection to factor sensitivities

In the absence of a model for the term structure of interest rates, it was noted above that, potentially, sensitivities with respect to *all* directions $v \in \mathscr{A}$ must be considered, and this is clearly not feasible. If we wish to limit the space of directions for forward rate curve movements, then we must assume a term structure model that admits a FDR in the sense of Björk and Svensson (2001) and Filipović and Teichmann (2004). We provide a brief, and somewhat non-rigorous, review of the results since they are crucial in establishing the directions along which interest rate sensitivities should be considered.

Let $n \in \mathbb{N}$ and let $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\in\mathbb{R}_+}, \mathbb{Q})$ be a probability basis satisfying the usual conditions, where $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ is generated by a standard n-dimensional $(\mathscr{F}_t, \mathbb{Q})$-Wiener process $w_t = (w_t^1, \ldots, w_t^n)$. Suppose that the admissible forward rate curves belong to a suitable subspace $\mathscr{H} \subset \mathscr{A}$. Moreover, assume that $\mathbb{Q}$ is an equivalent martingale measure for a term structure model in which the forward rate curve, r_t, evolves according to the stochastic differential equation (sde)

$$\mathrm{d}r_t(x) = \left[\frac{\partial r_t(x)}{\partial x} + \sigma(r_t)H\sigma(r_t) - \frac{1}{2}\sum_{i=1}^{n} \partial_{r_t}\sigma_i(\sigma_i(r_t))\right]\mathrm{d}t + \sigma(r_t) \circ \mathrm{d}w_t \quad (13)$$

with $r_0 = r^\star \in \mathscr{H}$, where $\sigma = (\sigma_1, \ldots, \sigma_n)\colon \mathscr{H} \to \mathscr{H}^n$ is suitably regular for the above equation to admit a continuous local weak solution, $\partial_{r_t}\sigma_i(\sigma_i(r_t))$ is the Fréchet derivative of σ_i at r_t in the direction $\sigma_i(r_t)$,

$$H\sigma(r_t, x) = \int_0^x \sigma(r_t, u)\,\mathrm{d}u, \quad (14)$$

and $\circ$ denotes Stratonovich differential. Let $\mu^\circ\colon \mathscr{H} \to \mathscr{H}$ be the Stratonovich forward rate drift in (13), so that

$$\mu^\circ(r_t) = \frac{\partial r_t(x)}{\partial x} + \sigma(r_t)H\sigma(r_t) - \frac{1}{2}\sum_{i=1}^{n} \partial_{r_t}\sigma_i(\sigma_i(r_t)). \quad (15)$$

For the technical details, including the definition of $\mathscr{H}$ and its properties, refer to Filipović (2001), Björk and Svensson (2001), and Filipović and Teichmann (2004).

The term structure model defined by (13) is said to admit a d-dimensional realization about $r^\star \in \mathscr{H}$ if there exists an open $U \subset \mathbb{R}^d$, a U-valued process,

z_t, satisfying an sde of the form

$$\mathrm{d}z_t = \mu^z(z_t)\mathrm{d}t + \sigma^z(z_t)\,\mathrm{d}w_t \tag{16}$$

with $z_0 = z^* \in U$, a regular submanifold map $G\colon U \to \mathscr{H}$, and a strictly positive stopping time τ such that $r_t = G(z_t)$ for $0 \leq t \leq \tau$.

For any subset $\mathscr{U} \subset \mathscr{H}$, and smooth vector fields $\alpha, \beta\colon \mathscr{U} \to \mathscr{H}$, the Lie product, $[\alpha, \beta]$, is defined as the smooth vector field on $\mathscr{U}$ given by

$$[\alpha, \beta](r) = \partial_r\alpha(\beta(r)) - \partial_r\beta(\alpha(r)). \tag{17}$$

If the model defined by (13) admits a d-dimensional realization about $r^\star$, then the local flows starting at $r^\star$ generate a d-dimensional submanifold $\mathscr{M} \subset \mathscr{H}$ and the tangent space, $T_r(\mathscr{M})$, to $r \in \mathscr{M}$ is the Lie algebra generated by the vector fields $\mu^\circ(r), \sigma_1(r), \sigma_2(r), \ldots, \sigma_n(r)$, so that

$$T_r(\mathscr{M}) = \langle \mu^\circ(r), \sigma_1(r), \sigma_2(r), \ldots, \sigma_n(r)\rangle_{\mathrm{LA}} \tag{18}$$

in the notation of Björk and Svensson (2001). In particular, if r is the current forward rate curve, then the directions in which it can evolve are given by the vector fields in $T_r(\mathscr{M})$, and since it is finite dimensional, it suffices to consider only the directions corresponding to a basis of $T_r(\mathscr{M})$ in assessing the risks associated with $\phi \in C^2(\mathscr{M})$ at r. We note that this is *not* equivalent to the *naïve* hedging strategy which only considers movements in the volatility directions as demonstrated below.

Example 1 Consider the Ritchken and Sankarasubramanian (1995) model corresponding to $n = 1$ and

$$\sigma(r, x) = h(\mathrm{ev}_0(r))\,\mathrm{e}^{-\kappa x}, \tag{19}$$

where $\kappa \in \mathbb{R}_+$, $h\colon \mathbb{R} \to \mathbb{R}$ is C^1, and $\mathrm{ev}_0\colon \mathscr{H} \to \mathbb{R}$ is the evaluation map at $x = 0$. It is easily shown that if $r^\star \in \mathscr{H}$ and $r^\star \notin \mathbb{R}\mathrm{e}^{-\kappa x} + \mathbb{R}\mathrm{e}^{-2\kappa x}$, then the resulting model with initial curve $r^\star$ admits a three-dimensional realization and

$$T_r(\mathscr{M}) = \mathbb{R}\frac{\partial r}{\partial x} + \mathbb{R}\mathrm{e}^{-\kappa x} + \mathbb{R}\mathrm{e}^{-2\kappa x}. \tag{20}$$

That is, the local evolutions of the forward rate curve, r, are limited to the directions $\partial r/\partial x$, $\mathrm{e}^{-\kappa x}$, $\mathrm{e}^{-2\kappa x}$ and their linear combinations. It should be noted that if $r^\star \in \mathbb{R}\mathrm{e}^{-\kappa x} + \mathbb{R}\mathrm{e}^{-2\kappa x}$, then we have the degenerate situation in which $\mathscr{M} \subset \mathbb{R}\mathrm{e}^{-\kappa x} + \mathbb{R}\mathrm{e}^{-2\kappa x}$ and $T_r(\mathscr{M}) = \mathbb{R}\mathrm{e}^{-\kappa x} + \mathbb{R}\mathrm{e}^{-2\kappa x}$ for all $r \in \mathscr{M}$.

Let $\phi \in C^2(\mathscr{M})$ such that $\partial_r\phi$ and $\partial_r^2\phi$ are uniformly continuous on bounded sets. Then application of Itô's lemma, see Da Prato and Zabczyk (1992), Theorem 4.17, gives

$$\mathrm{d}\phi(r_t) = \mu^\phi(r_t)\mathrm{d}t + \partial_{r_t}\phi(\sigma(r_t))\mathrm{d}w_t \tag{21}$$

for suitable μ^{ϕ}. Now, a *naïve* hedging strategy for ϕ, based on the usual stock option hedging argument, may be to choose the appropriate number, ξ_t^1, of an x_1-maturity zero coupon bond, p_{x_1}, so that

$$\partial_{r_t}\phi(\sigma(r_t)) + \xi_t^1 \partial_{r_t} p_{x_1}(\sigma(r_t)) = 0. \tag{22}$$

This corresponds to hedging against the forward rate curve movements only in the direction of $\mathrm{e}^{-\kappa x}$, leaving the remaining two directions unhedged.

To obtain a delta neutral hedge against all possible movements in the forward rate curve, we require two additional zero coupon bonds, say p_{x_2} and p_{x_3}, where without loss of generality $x_1 < x_2 < x_3$, and compute $\xi_t^1, \xi_t^2, \xi_t^3$ so that

$$\begin{aligned}
\xi_t^1 \partial_{r_t} p_{x_1}(\partial r_t/\partial x) + \xi_t^2 \partial_{r_t} p_{x_2}(\partial r_t/\partial x) + \xi_t^3 \partial_{r_t} p_{x_3}(\partial r_t/\partial x) &= -\partial_{r_t}\phi(\partial r_t/\partial x),\\
\xi_t^1 \partial_{r_t} p_{x_1}(\mathrm{e}^{-\kappa x}) + \xi_t^2 \partial_{r_t} p_{x_2}(\mathrm{e}^{-\kappa x}) + \xi_t^3 \partial_{r_t} p_{x_3}(\mathrm{e}^{-\kappa x}) &= -\partial_{r_t}\phi(\mathrm{e}^{-\kappa x}),\\
\xi_t^1 \partial_{r_t} p_{x_1}(\mathrm{e}^{-2\kappa x}) + \xi_t^2 \partial_{r_t} p_{x_2}(\mathrm{e}^{-2\kappa x}) + \xi_t^3 \partial_{r_t} p_{x_3}(\mathrm{e}^{-2\kappa x}) &= -\partial_{r_t}\phi(\mathrm{e}^{-2\kappa x}).
\end{aligned}$$

This system of equations can be inverted for the ξ_t^i and provides a delta hedge against all possible movements in the forward rate curve.

It is remarked that, given the explicit knowledge of the FDR in the above example, it is unlikely that the naïve hedging strategy would ever be considered. We now turn to a more systematic investigation of the relationship between factor sensitivities in a model with an FDR and the delta hedging introduced in this paper.

Under a reasonable set of assumptions on the forward rate volatilities, it was shown in Filipović and Teichmann (2004) that a term structure model (13) with an FDR must necessarily be *affine*. That is, there exist smooth functions $g_i\colon \mathbb{R}_+ \to \mathbb{R}$ such that the map $G\colon \mathbb{R}^d \supset U \to \mathscr{H}$ is of the form

$$G(z_t) = g_0 + g \cdot z_t, \tag{23}$$

where $g = (g_1, \ldots, g_d)$. Applying Itô's lemma, once again, then implies that the dynamics of the forward rate curve, $r_t = G(z_t)$, is given by

$$\mathrm{d}r_t = \mathrm{d}G(z_t) = \mu^z(z_t) \cdot g(x)\,\mathrm{d}t + \sigma^z(z_t)^{\mathrm{T}} g(x) \cdot \mathrm{d}w_t, \tag{24}$$

where the superscript $^{\mathrm{T}}$ denotes matrix transpose, and it follows that the forward rate volatilities are of the form

$$\sigma_i(r_t) = \sigma_i(G(z_t)) = \sum_j \sigma_{ji}^z(z_t) g_j. \tag{25}$$

If $T_r(\mathscr{M})$ is the tangent space to $r \in \mathscr{M}$, then $T_r(\mathscr{M})$ is $(d+1)$-dimensional, allowing for the additional factor corresponding to the running time t. In fact, it follows from the results in Filipović and Teichmann (2004) that

$$T_r(\mathscr{M}) = \mathbb{R}\mu^\circ(r) + \sum_{i=1}^{d} \mathbb{R}g_i. \tag{26}$$

Hence, the directions in which sensitivities must be considered at $r \in \mathscr{M}$ are $\mu^\circ(r)$ and g_i for $1 \leq i \leq d$.

Now given any $\varphi \in C^2(\mathscr{M})$, we can identify φ with $\bar{\varphi} = \varphi \circ G\colon U \to \mathbb{R}$. Then hedging against factor movements would require that we compute $\partial\bar{\varphi}/\partial z_t^i$, and construct a portfolio to neutralize these sensitivities. But by the chain rule

$$\frac{\partial\bar{\varphi}(z)}{\partial z_t^i} = \frac{\partial\varphi(G(z))}{\partial z_t^i} = \partial_r\varphi(\partial_{z_t^i}G(z)) = \partial_r\varphi(g_i), \tag{27}$$

and so the partial derivatives of $\bar{\varphi}$ with respect to the factors are equal to the deltas of φ in the direction g_i. Hence, the standard approach of hedging against the factor movements is equivalent to delta hedging in the directions g_i. Note, however, the direction $\mu^\circ(r)$ is still *not* considered in the former approach.

Remark 1 Note that even if a direction $v \notin \langle\mu^\circ, \sigma_1, \ldots, \sigma_n\rangle_{\mathrm{LA}}$, the risk along v can be considered in the framework of this paper. This is particularly relevant if one wishes to consider directions that are commonly observed, but not possible under the assumed term structure model. For example, if directions inconsistent with the model are revealed by principal component analysis, then it would still be prudent to consider risks along those directions.

Before closing this section, we investigate the relationship between the directional duration defined in (11) and the stochastic duration introduced in Cox et al. (1979), and subsequently extended in Munk (1999). Firstly, consider an affine short rate model in which the x-maturity zero coupon bond price, p_x, is given by

$$p_x(r) = \mathrm{e}^{-G_0(x) - G_1(x)\mathrm{ev}_0(r)}, \tag{28}$$

where ev_0 is the evaluation at $x = 0$ as defined above. Note that $\mathrm{ev}_0\colon \mathscr{A} \to \mathbb{R}$ is linear and bounded, and so Gâteaux differentiable with $\partial_r\mathrm{ev}_0 = \mathrm{ev}_0$. Arguments similar to those in the proof of Corollary 1 show that p_x is also Gâteaux differentiable with

$$\partial_r p_x(v) = -p_x(r)G_1(x)\mathrm{ev}_0(v). \tag{29}$$

Comparison with (8) implies $I_x(v) = G_1(x)\mathrm{ev}_0(v)$, which upon differentiating with respect to x gives

$$v(x) = g_1(x)v(0), \tag{30}$$

where $g_1 = G_1'$. Note that for short rate models, $g_1(0) = 1$, so that $v = g_1$ satisfies the above equation.

Now, let $m \in \mathbb{N}$, $\boldsymbol{c} \in \mathbb{R}^m$, $\boldsymbol{x} \in \mathbb{R}_+^m$, and define $\phi_{\boldsymbol{c},\boldsymbol{x}} \in C^1(\mathscr{A})$ as in (9). Then the stochastic duration of $\phi_{\boldsymbol{c},\boldsymbol{x}}$ at $r \in \mathscr{A}$ is defined in Cox et al. (1979) by

$$D_{\text{CIR}}\phi_{\boldsymbol{c},\boldsymbol{x}}(r) = G_1^{-1}\left[\frac{1}{\phi_{\boldsymbol{c},\boldsymbol{x}}(r)}\sum_{i=1}^m c_i p_{x_i}(r)\, G_1(x_i)\right]. \tag{31}$$

Proposition 2 *Let $\phi_{\boldsymbol{c},\boldsymbol{x}} \in C^1(\mathscr{A})$ be as defined above. Then*

$$D_{\text{CIR}}\phi_{\boldsymbol{c},\boldsymbol{x}}(r) = G_1^{-1}[D_r^1\phi_{\boldsymbol{c},\boldsymbol{x}}(G_1')]. \tag{32}$$

Proof Follows immediately from the remark following (30). □

Consistent with approaches that only consider sensitivities with respect to the factors, the stochastic duration ignores the sensitivity with respect to forward rate curve movements in the direction μ°. Although this is unnecessary in this case since μ° and G_1' are parallel, it is not the case in general. Moreover, although the motivation behind taking the inverse of G_1 in (31) is to obtain "a proxy for basis risk of coupon bonds with the units of time," according to Cox et al. (1979), it seems that one loses, in some sense, its properties as a measure of interest rate sensitivity in the process.

Finally, consider the general dynamics of zero coupon bonds

$$\mathrm{d}p_x(r_t) = p_x(r_t)\left[\mu^{p_x}(r_t)\mathrm{d}t + \sigma^{p_x}(r_t)\cdot \mathrm{d}w_t\right], \tag{33}$$

and note that

$$\mathrm{d}\phi_{\boldsymbol{c},\boldsymbol{x}}(r_t) = \phi_{\boldsymbol{c},\boldsymbol{x}}(r_t)\left[\sum_{i=1}^m \xi_t^i \mu^{x_i}(r_t)\mathrm{d}t + \left(\sum_{i=1}^m \xi_t^i \sigma^{p_{x_i}}(r_t)\right)\cdot \mathrm{d}w_t\right], \tag{34}$$

where

$$\xi_t^i = \frac{c_i p_{x_i}(r_t)}{\sum_j c_j p_{x_j}(r_t)}. \tag{35}$$

In Munk (1999), the stochastic duration of $\phi_{\boldsymbol{c},\boldsymbol{x}}$ is defined as the "maturity of the zero coupon bond having the same relative volatility as" $\phi_{\boldsymbol{c},\boldsymbol{x}}$. That is, the quantity, $D_{\text{Mun}}\phi_{\boldsymbol{c},\boldsymbol{x}}(r_t)$, that solves the equation

$$\left|\sigma^{p_{D_{\text{Mun}}\phi_{\boldsymbol{c},\boldsymbol{x}}(r_t)}}(r_t)\right|^2 = \left|\sum_{i=1}^m \xi_t^i \sigma^{p_{x_i}}(r_t)\right|^2. \tag{36}$$

For short rate models, it reduces to $D_{\text{CIR}}\phi_{\boldsymbol{c},\boldsymbol{x}}(r_t)$. Unfortunately, $D_{\text{Mun}}\phi_{\boldsymbol{c},\boldsymbol{x}}(r_t)$ does not always exist, and its significance as a measure of risk sensitivity is somewhat unclear. Moreover, as with previous cases, it does not consider all possible directions in which the forward rate curve may move. The situation is even more serious here, since the general definition of $D_{\text{Mun}}\phi_{\boldsymbol{c},\boldsymbol{x}}(r_t)$ in Munk (1999) does not assume that the model admits an FDR, in which case the forward rate curve may, potentially, move in all directions $v \in \mathscr{A}$.

5 Delta and gamma in general

For the purposes of pricing and hedging interest rate derivatives, we would like there to be a subset $\mathscr{U} \subset \mathscr{A}$ such that if the forward rate curves generated by a model belong to $\mathscr{U}$, then for any reasonable $\varphi \colon \mathscr{U} \to \mathbb{R}$ and a stopping time τ we have

$$\mathbb{E}^{\mathbb{Q}}\left[\mathrm{e}^{-\int_t^\tau \mathrm{ev}_0(r_u)\,\mathrm{d}u}\varphi(r_\tau)\middle|\mathscr{F}_t\right] = u^\varphi(t, r_t) \tag{37}$$

for some $u^\varphi \colon \mathbb{R}_+ \times \mathscr{U} \to \mathbb{R}$. Moreover, it would be desirable for u^φ to be (perhaps twice) differentiable with respect to r_t, since then it would not only be possible to price, but also to compute the directional delta and the gamma for the derivative with payoff φ. Unfortunately, the literature dealing with this problem is very limited, with Goldys and Musiela (2001) being the only reference known to the author. It is shown there that if $\mathscr{U} = L^2(\mathbb{R}_+)$ and either r_t is generated by a Gaussian HJM model or $\varphi \in C^2(\mathscr{U})$, along with some other technical conditions, then (37) holds. These conditions, however, are too restrictive for most term structure models and interest rate derivatives. For example, models with level dependent forward rate volatilities, such as the Cox et al. (1985) model, and many derivatives, such as European calls and puts on zero coupon bonds whose payoff functions, φ, are non-differentiable with respect to the forward rate curve, do not fall within this framework. It would be useful to obtain less restrictive conditions under which (37) holds.

6 A comparison of hedging strategies

In this section, we consider the problem of hedging the price movements in a coupon bond with a set of zero coupon bonds under the Ritchken and Sankarasubramanian (1995) model in (19) with $h(u) = \sigma_0\sqrt{u}$. For this model, it is well-known that the forward rate curve is given by

$$r_t(x) = r_0(t+x) + \mathrm{e}^{-\kappa x} z_t^1 + \frac{1}{\kappa}\mathrm{e}^{-\kappa x}(1 - \mathrm{e}^{-\kappa x}) z_t^2, \tag{38}$$

where $z_0^1 = 0 = z_0^2$, and z_t^1 and z_t^2 satisfy the stochastic differential equations

$$\mathrm{d}z_t^1 = \left[\partial_t r_0(t) - \kappa z_t^1 + z_t^2\right]\mathrm{d}t + \sigma_0\sqrt{z_t^1 + r_0(t)}\,\mathrm{d}w_t, \tag{39}$$

$$\mathrm{d}z_t^2 = (\sigma_0^2(z_t^1 + r_0(t)) - 2\kappa z_t^2)\mathrm{d}t. \tag{40}$$

The bond prices are given by

$$p_t(x) = \frac{p_0(t+x)}{p_0(t)}\, \mathrm{e}^{-\alpha(x) z_t^1 - \frac{1}{2}\alpha^2(x) z_t^2}, \tag{41}$$

where $\alpha(x) = (1 - \mathrm{e}^{-\kappa x})/\kappa$. For the numerical comparison, we take $\kappa = 0.2339$, $\sigma_0 = 0.0854$, and initial curve $r_0(x) = 0.07715 - 0.01\mathrm{e}^{-3.5\kappa x}$. The parameters κ and σ_0 are taken from Chan et al. (1992) for the Cox et al. (1985) model, and the initial curve is taken to be upward sloping with the short rate equal to the average short rate from loc. cit.

We use three different hedging approaches, viz. those based on Macaulay duration, factor sensitivities, and directional delta. To ensure a fair comparison, we use the same number of zero coupon bonds for each approach. As is customary, we set the initial values of the hedge portfolios equal to the value of the asset being hedged. For the Macaulay duration approach, we also matched the duration, convexity, and the third order equivalent. For the factor sensitivities approach we matched the first derivatives with respect to z_t^1 and z_t^2, and the second order derivative with respect to z_t^1 and z_t^2. For the directional delta approach, we matched the deltas along the directions $\partial_x r_t(x)$, $\mathrm{e}^{-\kappa x}$, and $\mathrm{e}^{-2\kappa x}$.

The asset being hedged is a 3-year semi-annual coupon bond with coupon rate 6%, and the assets used for the hedge are the zero coupon bonds with maturities $\frac{1}{2}k$ years, where $1 \le k \le 4$. The face values of all the bonds are assumed to be 100. The optimal hedge for each approach is formed and held for 1 week. Using Monte Carlo with 10 time steps, we have simulated 100,000 terminal values of z_t^1 and z_t^2. At the end of the hedge period, the values of the hedge portfolios were computed. The summary statistics for the hedging errors are given in Table 1. Note that although the directional delta approach has the smallest average error, the factor sensitivities approach has the smallest variance in the errors. It should also be noted that the relative performance of these two approaches are sensitive to the initial forward rate curve.

A scatter plot of the hedging errors for the factor sensitivities and the directional delta approaches are shown in Fig. 1. It is evident from the figure that for small movements in the forward rate curve, the directional delta approach produces smaller hedging errors. The size of the hedging errors, however, increases faster for the directional delta approach as the forward rate curve undergoes larger movements. This is not surprising since, due to the special form of (41) for the bond price in this case, two of the three second-order derivatives with respect to the factors are hedged in the factor sensitivities approach.

Table 1 Summary statistics for hedging errors

	Mean	SD	Min	Max
Macaulay	-0.004773	0.093070	-0.403870	0.395040
Factors	2.4838×10^{-7}	1.8585×10^{-9}	2.4453×10^{-7}	3.2626×10^{-7}
Directional	1.3012×10^{-7}	4.2643×10^{-8}	-9.9826×10^{-7}	1.2768×10^{-6}

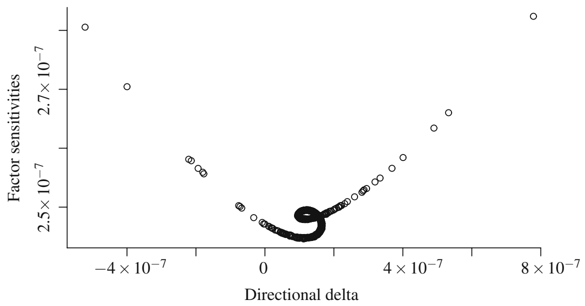

Fig. 1 Scatter plot of hedging errors

7 Conclusion

In this paper, we introduced sensitivity measures for interest rate derivatives that naturally generalize the delta and the gamma for stock options. These measures capture the sensitivities along specific directions in which the forward rate curve may evolve, and include the traditional measures, such as Macaulay and stochastic durations, as special cases. Sensitivity analysis with respect to principal components is consistent with this approach, as is the sensitivity analysis with respect to factors when a finite factor model for the term structure of interest rates is assumed.

It was shown that most existing hedging approaches for interest rate derivatives are inadequate in that they only consider risks with respect to a limited set of directions along which the forward rate curve can evolve. An example using the Ritchken and Sankarasubramanian (1995) model revealed that a naïve adaptation of delta hedging for stock options only provides hedge against movements along the direction of forward rate volatilities and leaves unhedged risks along two remaining directions. This was supported by a numerical comparison of the various hedging approaches in which the directional duration approach was shown to provide the smallest hedging errors under small movements in the forward rate curve.

Acknowledgment The author thanks the anonymous referee for helpful comments and suggestions.

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